1. Consider the matrix S with diagonalization

$$S = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$$

   (a) What are the eigenvectors and eigenvalues of S?

   (b) Explain how you know S is symmetric from its diagonalization, without computing S.

   (c) Find the determinant of S.

   (d) Does the transformation S preserve or reverse orientation, or neither of these?

   (e) What is the volume of the image of the unit cube after being transformed by S?

   (f) Is S invertible?

   (g) Find a diagonalization for S^3.

   (h) Let $\mathbf{x} = (2, 8, 7)$. Compute $S^3\mathbf{x}$. [Hint: $(2, 8, 7) = 3(2, 2, 1) + 2(-2, 1, 2)$.]

   (i) Find a matrix A such that $S = A^T A$, or explain why it is not possible.

Solution:

   (a) S has eigenvectors $(2, 2, 1)$, $(-2, 1, 2)$, and $(1, -2, 2)$, with eigenvalues 2, 1, and -3.

   (b) The diagonalization of S has the form $Q\Lambda Q^T$, where Q is an orthogonal matrix and Λ is diagonal, so A is symmetric.

   (c) Since $S = Q\Lambda Q^T$, and Q and $Q^T = Q^{-1}$ have determinants that are reciprocals of each other (In fact, since they are orthogonal, they are both 1 or both -1),

$$\det S = \det Q\Lambda Q^T = \det Q \det \Lambda \det Q^T = \frac{\det Q \det \Lambda}{\det Q} = \det \Lambda.$$

   Since the determinant of a diagonal matrix is the product of the entries on the diagonal, $\det S = \det \lambda = (2)(1)(-3) = -6$.

   (d) Since $\det S < 0$, the transformation S reverses orientation.

   (e) Since $|\det S| = 6$, the volume of the image of the unit cube after being transformed by S is 6.

   (f) Since $\det S \neq 0$, S is invertible.

   (g)

$$S^3 = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 2^3 & 0 & 0 \\ 0 & 1^3 & 0 \\ 0 & 0 & (-3)^3 \end{pmatrix} \quad \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$$

(h) Since $(2,2,1)$ and $(-2,1,2)$ are eigenvectors for S with eigenvalues 2 and 1,

$$\begin{aligned} S^3\mathbf{x} &= S^3(2,8,7) \\ &= S^3(3(2,2,1)+2(-2,1,2)) \\ &= 3S^3(2,2,1)+2S^3(-2,1,2) \\ &= 3(2^3)(2,2,1)+2(1^3)(-2,1,2) \\ &= (44,50,28). \end{aligned}$$

(i) Since S has a negative eigenvalue, it is not positive semi-definite, and thus there is no matrix A such that $S = A^TA$.

2. Let A be an $m \times n$ matrix with linearly independent columns.

(a) Give a formula for the projection matrix P such that for any vector $\mathbf{b} \in \mathbb{R}^m$, $P\mathbf{b}$ is the projection of $\mathbf{b}$ onto $C(A)$.

(b) How many rows and columns does P have?

(c) Prove that $P^2 = P$.

(d) If a square matrix B has eigenvector $\mathbf{x}$ with eigenvalue λ, show that $\mathbf{x}$ is also an eigenvector of B^2. What is its eigenvalue?

(e) Use the previous two parts to explain why the only possible eigenvalues for P are 0 and 1.

(f) In terms of m and n, what are the algebraic and geometric multiplicities of the eigenvalues 0 and 1 for P?

Solution:

(a) $P = A(A^TA)^{-1}A^T$.

(b) Since A is $m \times n$, A^T is $n \times m$, so both A^TA and $(A^TA)^{-1}$ are $n \times n$, and the product $A(A^TA)^{-1}A^T$ is $m \times m$.

(c) Since $(A^TA)^{-1}(A^TA) = I$,

$$P^2 = (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T) = A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P.$$

(d) Since

$$B^2\mathbf{x} = B(B\mathbf{x}) = B(\lambda\mathbf{x}) = \lambda B\mathbf{x} = \lambda^2\mathbf{x},$$

$\mathbf{x}$ is an eigenvector of B^2 with eigenvalue λ^2.

(e) Since $P^2 = P$, if $\mathbf{x}$ is an eigenvector for P with eigenvalue λ then $\mathbf{x}$ is an eigenvector for P^2 with eigenvalue λ. On the other hand, $\mathbf{x}$ is an eigenvector for P^2 with eigenvalue λ^2. Thus $\lambda = \lambda^2$, or $0 = \lambda^2 - \lambda = \lambda(\lambda - 1)$. Thus λ must equal 0 or 1. (I.e. the only real numbers equal to their squares are 0 and 1.)

(f) Since the columns of A are linearly independent, the rank of A is n, and $\dim C(A) = n$. Since $\mathbf{x} \in C(A)$ if and only if $P\mathbf{x} = \mathbf{x}$, we can find n independent vectors with eigenvalue 1: a basis for the column space. Since $A^T\mathbf{x} = \mathbf{0}$ implies $P\mathbf{x} = A(A^TA)^{-1}A^T\mathbf{x} = A(A^TA)^{-1}\mathbf{0} = \mathbf{0}$, all vectors in the left null space of A have eigenvalue 0. So we can find $m-n$ independent vectors with eigenvalue 0: a basis for the left null space.

3. Let A be a 4×3 matrix with singular vectors $\mathbf{v}_1 = (1/2, \sqrt{1/2}, 1/2)$ and $\mathbf{v}_2 = (1/2, -\sqrt{1/2}, 1/2)$ and singular values $\sigma_1 = 4$ and $\sigma_2 = 3$. These are all of the singular values.

   (a) What is the rank of A?

   (b) Find the dimensions of the four fundamental subspaces for A.

   (c) Find the following lengths: $\|A\mathbf{v}_1\|$, and $\|A\mathbf{v}_2\|$

   (d) Write the matrices V_2 and Σ_2 for the rank 2 format singular value decomposition for A, and write an expression for A^TA in terms of these matrices. [Write out all of the matrices numerically, but do not compute A^TA for this part of the problem, or any subsequent parts.]

   (e) What are the eigenvalues of A^TA?

   (f) What is the determinant of A^TA?

   (g) Let $\mathbf{e}_1 = (1, 0, 0)$. Find the projection of $\mathbf{e}_1$ onto $\mathbf{v}_1$ and $\mathbf{v}_2$.

   (h) Let $\mathbf{x}$ be the projection of $\mathbf{e}_1$ onto $C(A^T)$. Use your answer to Part (g) to write $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

   (i) Find the following lengths: $\|A\mathbf{x}\|$, and $\|A\mathbf{e}_1\|$.

   Solution:

   (a) Because A has two singular values, the rank of A is 2.

   (b) $\dim C(A^T) = 2$, $\dim N(A) = 2$, $\dim C(A) = 2$, $\dim N(A^T) = 1$.

   (c) Since $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$,
   - $\|A\mathbf{v}_1\| = \|4\mathbf{u}_1\| = 4\|\mathbf{u}_1\| = 4 \cdot 1 = 4$
   - $\|A\mathbf{v}_2\| = \|3\mathbf{u}_2\| = 3\|\mathbf{u}_2\| = 3 \cdot 1 = 3$

   (d)
   $$V_2 = \begin{pmatrix} 1/2 & 1/2 \\ \sqrt{1/2} & -\sqrt{1/2} \\ 1/2 & 1/2 \end{pmatrix} \qquad \text{and} \qquad \Sigma_2 = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}.$$

   Since
   $$A^TA = (V_2\Sigma_2^TU_2^T)(U_2\Sigma_2V_2^T) = V_2\Sigma_2^T(U_2^TU)\Sigma_2V_2^T = V_2\Sigma_2^T\Sigma_2V_2^T,$$

$$
\begin{aligned}
A^TA &= V_2\Sigma_2^T\Sigma_2V_2^T \\
&= \begin{pmatrix} 1/2 & 1/2 \\ \sqrt{1/2} & -\sqrt{1/2} \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1/2 & \sqrt{1/2} & 1/2 \\ 1/2 & -\sqrt{1/2} & 1/2 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & 1/2 \\ \sqrt{1/2} & -\sqrt{1/2} \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 16 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1/2 & \sqrt{1/2} & 1/2 \\ 1/2 & -\sqrt{1/2} & 1/2 \end{pmatrix}.
\end{aligned}
$$

(e) Since A^TA is a positive semidefinite matrix, and the singular values are the square roots of the positive eigenvalues of A^TA, the eigenvalues of A^TA are 16, 9, and 0.

(f) The determinant of A^TA is the product of its eigenvalues, $\det(A^TA) = 0$.

(g)
- $\text{proj}_{\mathbf{v}_1}\mathbf{e}_1 = (\mathbf{v}_1^T\mathbf{e}_1)\mathbf{v}_1 = (1/2)\mathbf{v}_1 = (1/4, \sqrt{1/8}, 1/4)$.
- $\text{proj}_{\mathbf{v}_2}\mathbf{e}_1 = (\mathbf{v}_2^T\mathbf{e}_1)\mathbf{v}_2 = (1/2)\mathbf{v}_2 = (1/4, -\sqrt{1/8}, 1/4)$.

(h) Since $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis for $C(A^T)$, the projection of $\mathbf{e}_1$ onto $C(A^T)$ is the sum of its projections onto $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus

$$\mathbf{x} = \text{proj}_{\mathbf{v}_1}\mathbf{e}_1 + \text{proj}_{\mathbf{v}_2}\mathbf{e}_1 = (1/2)\mathbf{v}_1 + (1/2)\mathbf{v}_2.$$

(i) Since $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$,

- Since $\mathbf{x} = (1/2)\mathbf{v}_1 + (1/2)\mathbf{v}_2$,

$$
\begin{aligned}
\|A\mathbf{x}\| &= \|A((1/2)\mathbf{v}_1 + (1/2)\mathbf{v}_2)\| \\
&= \|(1/2)A\mathbf{v}_1 + (1/2)A\mathbf{v}_2\| \\
&= \|2\mathbf{u}_1 + (3/2)\mathbf{u}_2\| \\
&= \sqrt{2^2 + (3/2)^2} \\
&= 5/2.
\end{aligned}
$$

- Since $\mathbf{e}_1 = (1/2)\mathbf{v}_1 + (1/2)\mathbf{v}_2 + c_3\mathbf{v}_3$, where $\mathbf{v}_3$ is the third singular vector (and is a basis vector for the 1-dimensional null space of A),

$$
\begin{aligned}
\|A\mathbf{e}_1\| &= \|A((1/2)\mathbf{v}_1 + (1/2)\mathbf{v}_2 + c_3\mathbf{v}_3)\| \\
&= \|(1/2)A\mathbf{v}_1 + (1/2)A\mathbf{v}_2 + c_3A\mathbf{v}_3\| \\
&= \|2\mathbf{u}_1 + (3/2)\mathbf{u}_2 + \mathbf{0}\| \\
&= \sqrt{2^2 + (3/2)^2} \\
&= 5/2.
\end{aligned}
$$